CLASSIFICATION OF STABLE MINIMAL SURFACES BOUNDED BY JORDAN CURVES IN CLOSE PLANES

ROSANNA PEARLSTEIN

Abstract

We study compact stable embedded minimal surfaces whose boundary is given by two collections of closed smooth Jordan curves in close planes of Euclidean Three-Space. Our main result is a classification of these minimal surfaces, under certain natural geometric asymptotic constraints, in terms of certain associated varifolds which can be enumerated explicitely. One consequence of this result is the uniqueness of the area-minimizing examples. Another is the asymptotic non-existence of stable compact embedded minimal surfaces of positive genus bounded by two convex curves in parallel planes.

Acknowledgments

The author wishes to thank her thesis advisor, professor Bill Meeks, for suggesting this problem to her, and for many helpful conversations.

1.0 Introduction

At the Princeton Bicentennial Conference in 1946, Tibor Radó posed the problem to estimate the number of compact minimal surfaces bounded by one or more given Jordan curves in Euclidean three-space. We will answer this question of Radó under the assumption that the surfaces are embedded and stable, and their boundary curves satisfy a natural asymptotic geometric constraint. We now describe our results and methods in more detail. Throughout this paper, $\mathcal{A}$ and $\mathcal{B}$ will each denote a collection of disjoint closed smooth Jordan curves in the plane $P_0 : z = 0$ of $\mathbb{R}^3$. We make the assumption that $\mathcal{A}$ and $\mathcal{B}$ intersect transversely in a finite collection of points, which we call *crossing points*. We denote by $\mathcal{B}(t)$ the vertical translation of $\mathcal{B}$ to the plane $P_t : z = t$. Consider the immersed 1-cycle $Z = \mathcal{A} \cup \mathcal{B}$ in P_0, and let $\mathcal{C}(\mathcal{A}, \mathcal{B})$ denote the collection of the closures of the bounded components in $P_0 \backslash Z$. Let $\mathcal{V}(\mathcal{A}, \mathcal{B})$ denote the (finite) collection of integer

multiplicity varifolds that can be represented by nonnegative integer multiple sums of the components in $\mathcal{C}(\mathcal{A},\mathcal{B})$, where the multiplicity changes by *one* in passing from any component to an adjacent one, and at least one of the four components meeting at a crossing point has multiplicity zero (note that as a consequence these varifolds have Z as their $\mathbb{Z}_2$-boundary). Let $\mathcal{S}(t)$ be the collection of compact stable embedded minimal surfaces bounded by $\mathcal{A}\cup\mathcal{B}(t)$. Our main theorem is:

Theorem 1.1 (Main Theorem) *For values of t sufficiently small, one can naturally associate to a varifold V in $\mathcal{V}(\mathcal{A},\mathcal{B})$ a unique compact stable embedded minimal surface $\Sigma(V,t)$ in $\mathcal{S}(t)$ bounded by $\mathcal{A}\cup\mathcal{B}(t)$. Furthermore, this association between varifolds in $\mathcal{V}(\mathcal{A},\mathcal{B})$ and surfaces in $\mathcal{S}(t)$ is a one-to-one correspondence.*

In order to discuss further the geometry and topology of the stable minimal surface $\Sigma(V,t)$ associated to a varifold $V\in\mathcal{V}(\mathcal{A},\mathcal{B})$, we need to give a few definitions. Note that around each crossing point, Z divides a sufficiently small neighborhood D in four connected components; we order them according to the counterclockwise orientation of D, so that the multiplicity of the first component is zero. Now fix a $V\in\mathcal{V}(\mathcal{A},\mathcal{B})$. Our assumptions on V imply that only the multiplicities $(0,1,0,1)$ and $(0,1,2,1)$ can occur at the crossing points. We let v_1 be the number of crossing points with multiplicity $(0,1,0,1)$ in $\mathcal{A}\cap\mathcal{B}$, let v_2 be the number of crossing points with multiplicity $(0,1,2,1)$ in $\mathcal{A}\cap\mathcal{B}$, let f_1 be the number of components of V with multiplicity one, let f_2 be the number of components of V with multiplicity two, let e_1 be the number of multiplicity one connected components of $Z\setminus\{crossing\ \ points\}$, let e_2 be the number of multiplicity two connected components of $Z\setminus\{crossing\ \ points\}$. Then we have the following result.

Theorem 1.2 *Let $V\in\mathcal{V}(\mathcal{A},\mathcal{B})$, and for t sufficiently small let $\Sigma(V,t)$ be the surface given in the statement of theorem* 1.1. *Then the Euler characteristic of $\Sigma(V,t)$ is equal to $(v_1+2v_2)-(e_1+2e_2)+(f_1+2f_2)$.*

It follows that since there is exactly one varifold $V_0\in\mathcal{V}(\mathcal{A},\mathcal{B})$ having least area (namely no multiplicity 2 components), there is exactly one $\Sigma(V_0,t)$ in $\mathcal{S}(t)$ with Euler characteristic equal to $N-l+f$, where N is the number of crossing points, l the number of connected components of $Z\setminus crossing\ points$, f is the number of multiplicity one components of V. This surface $\Sigma(V_0,t)$ is the unique surface of least area in $\mathcal{S}(t)$.

The first step in our proof of theorem 1.1 is to show that, for any fixed varifold V, there exists a compact stable embedded minimal surface bounded

by $\mathcal{A}\cup\mathcal{B}(t)$ corresponding to V, when t is sufficiently small. More precisely, we prove the following theorem, which follows from an existence result proven by W. Meeks and S. T. Yau [21].

Theorem 1.3 *Let $\mathcal{A}$ and $\mathcal{B}(t)$ be two collections of disjoint closed smooth Jordan curves in the planes P_0 and P_t respectively, such that $\mathcal{A}$ intersects $\mathcal{B} = \mathcal{B}(0)$ transversely in a finite number of points. Fix a varifold V in $\mathcal{V}(\mathcal{A}, \mathcal{B})$. Then, for every sufficiently small t, there exists a compact stable embedded minimal surface $\Sigma(t)$ with $\partial(\Sigma(t)) = \mathcal{A}\cup\mathcal{B}(t)$, and such that $\Sigma(t)$ is "close to" V, in the sense that the family of minimal surfaces $\{\Sigma(t)\}_t$ converges to V as t approaches 0.*

Next we describe geometrically the stable minimal surfaces obtained in the previous theorem, extending a previous result by W. Rossman [31].

Theorem 1.4 *Let $\mathcal{A}\cup\mathcal{B}(t)$ be the boundary of compact stable embedded minimal surfaces $\Sigma(t)$ arising in theorem 1.3. Then, for t sufficiently close to 0, the surfaces $\Sigma(t)$ have the following properties:*

1. *In the complement of small vertical cylindrical neighborhoods $N(i)$ of the crossing points, the components of $\Sigma(t)\setminus\bigcup N(i)$ are graphs over their projections to the plane P_0.*

2. *In $N(i)$, $\Sigma(t)$ is either approximately helicoidal, or is the union of two graphs over the plane P_0.*

Next we prove that if a minimal surface $\Sigma(t)$ possesses a description as in theorem 1.4, then such minimal surface must be stable, for t sufficiently small.

Theorem 1.5 *Let $\Sigma(t)$ be a minimal surface with boundary $\mathcal{A}\cup\mathcal{B}(t)$. Suppose that $\Sigma(t)$ is described as in theorem 1.4. Then, for t sufficiently close to 0, $\Sigma(t)$ is stable.*

The proof of the above theorem involves some technical lemmas; we give here a sketch of the proof of theorem 1.5, referring the reader to section 4 for details. A key step of the proof is the following observation: let Σ be a minimal surface, given by the union $\Sigma_1\cup S\cup\Sigma_2$, and suppose that the first eigenvalue of the Jacobi operator on Σ_1 and Σ_2 is strictly positive, and S is a simply connected sufficiently flat piece, in the sense that the diameter of its Gaussian image is very small. Then the first eigenvalue of the Jacobi operator on the minimal surface Σ is also strictly positive. Our

hypotheses allow us to apply some techniques developed by Kapouleas in [12], and used in his construction of constant mean curvature surfaces in $\mathbb{R}^3$. These results imply that if one glues two stable pieces E_1 and E_2 by a sufficiently flat bridge S, then no Jacobi vector fields arise on $E_1 \cup S \cup E_2$, since none arise from E_1 or E_2, and no new ones arise from gluing S. In the case corresponding to a surface $\Sigma(V,t)$ in theorem 1.5, we start with E_1 and E_2 each being a small helicoidal minimal surfaces near a crossing point, and S being one of the flat components of $\Sigma(t) \backslash \bigcup N(i)$ adjacent to E_1, and E_2, and is the region that glues E_1 with E_2. We can do this by the description of $\Sigma(V,t)$ given in theorem 1.4. A proof by induction on the number of helicoidal components shows that $\Sigma(V,t)$ is stable. This completes our brief sketch of theorem 1.5.

Remark 1.1 *This inspires a* Gauss-map bridge principle*, which will be discussed more extensively elsewhere.*

We now give a brief sketch of the proof of our Main Theorem. By theorem 1.3, we can associate to each varifold $V \in \mathcal{V}(\mathcal{A}, \mathcal{B})$ a minimal surface $\Sigma(V,t)$, for t sufficiently small. The Main Theorem states that $\Sigma(V,t)$ is eventually unique. We now give a sketch of the proof of the uniqueness. Suppose that there are two sequences of stable minimal surfaces $\Sigma_1(t_i)$ and $\Sigma_2(t_i)$ which have $\partial(\Sigma_1(t_i)) = \partial(\Sigma_2(t_i)) = \mathcal{A} \cup \mathcal{B}(t_i)$, where t_i approaches 0 as i tends to infinity. We show that if such a sequence of minimal surfaces existed, then there would be two other sequences of embedded stable minimal surfaces with boundary given by $\mathcal{A} \cup \mathcal{B}(t_i)$, and these surfaces could be taken so that their interiors would be disjoint. So we may assume that $\Sigma_1(t_i)$ and $\Sigma_2(t_i)$ are disjoint in their interior, for all t_i, and that both sequences converge to a fixed varifold V, as $t_i \to 0$. Theorem 1.4 implies that, for i large, the surfaces $\Sigma_1(t_i)$ and $\Sigma_2(t_i)$ bound a product region $R(i)$, where the interior angles are less than π, and the surfaces are ambient isotopic in $R(i)$. By a deep minimax theorem by Pitts and Rubinstein [30], generalized to the case of nonempty boundary by Jost [11], there would exist an unstable embedded minimal surface $\Sigma^*(t_i)$ in $R(i)$, such that $\partial(\Sigma^*(t_i)) = \partial(\Sigma_1(t_i)) = \partial(\Sigma_2(t_i)) = \mathcal{A} \cup \mathcal{B}(t_i)$. Because $\Sigma_1(t_i)$ and $\Sigma_2(t_i)$ are expressed as normal graphs over each other, and by the definition of minimax, we are able to show that, away from the crossing points, $\Sigma^*(t_i)$ is a very flat graph. Then Nitsche's 4π-theorem allows us to show that $\Sigma^*(t_i)$ is approximately helicoidal around the crossing points. Hence the unstable minimal surface $\Sigma^*(t_i)$ would possess the same geometric description as $\Sigma_1(t_i)$ and $\Sigma_2(t_i)$. Then theorem 1.5 implies that, for i sufficiently large, $\Sigma^*(t_i)$ would have to be stable. This produces a contradiction, and finishes the sketch of our proof of the Main Theorem.

We then give an upper bound on the number of compact embedded stable minimal surfaces bounded by two given collections of curves. More precisely, we show the following.

Corollary 1.1 *Once the limiting cycle* $Z = \mathcal{A}\cup\mathcal{B}$ *in the plane* P_0 *is given, the number of stable compact minimal surfaces* $\Sigma(t)$ *such that* $\partial(\Sigma(t))\rightarrow Z$ *as* $t\rightarrow 0$*, is bounded above by*

$$2^{f^i_-} + 2^{f^o_-},$$

for t *sufficiently close to* 1*, where* f^i_- *is the number of bounded components of* $P_0\backslash Z$ *inside Region*$(\mathcal{A})\cap$*Region*$(\mathcal{B})$*, and* f^o_- *is the number of bounded components of* $P_0\backslash Z$ *outside Region*$(\mathcal{A})\cup$*Region*$(\mathcal{B})$*.*

Finally, we observe how the above results support a conjecture made by W.H. Meeks in [14], about the nonexistence of positive genus compact embedded minimal surfaces bounded by two convex curves in parallel planes of $\mathbb{R}^3$.

2.0 Preliminary notions

In this section we recall some known facts about minimal surfaces.

2.1 Minimal surfaces are locally area-minimizing

There are several possible ways to define minimal surfaces in $\mathbb{R}^3$. A surface S in $\mathbb{R}^3$ is *minimal* if for every point p of S it is possible to find a neighborhood U_p of p, contained in S, which is the unique surface of least area among all surfaces with boundary ∂U_p. Notice that S could have infinite area, for instance, if S is a plane in $\mathbb{R}^3$. A minimal surface can also be defined as a surface for which every compact subdomain C is a critical point for the area functional among all surfaces having boundary equal to ∂C. From the point of view of local geometry, this is equivalent to the condition that the mean curvature H be identically zero.

Theorem 2.1 (Meusnier) *A regular surface* Σ *immersed in* $\mathbb{R}^3$ *is a critical point for the area functional if and only if the mean curvature vanishes identically on* Σ*.*

2.2 Stability: eigenvalues of the Laplacian

Recall that the Gauss map of an oriented surface in $\mathbb{R}^3$ assigns to each point of the surface the unit normal vector to the surface at that point, viewed as an element of the unit sphere S^2 in $\mathbb{R}^3$. We fix an orientation on S^2 by the inward pointing normal vector.

Theorem 2.2 (Christoffel) *Given a surface $\Sigma \hookrightarrow \mathbb{R}^3$, the Gauss map g is conformal or anti-conformal if and only if Σ is a minimal surface (or a sphere).*

Proof. The proof of this theorem follows from the definitions of mean curvature and Gauss map. A complete proof can be found in [36, p. 385, vol. 4], and the original proof in[5]. q.e.d.

Recall that a minimal surface is stable if, for each subdomain D the second derivative of area is positive for each smooth normal variation of D which is the identity on $\partial D = \Gamma$, and *unstable* if for some smooth normal variation of D which is the identity on $\partial D = \Gamma$ the second derivative of area is negative.

We now recall another characterization of stability. We start with the second variational formula (see for example [**?**]):

Theorem 2.3 (Second variational formula)

$$\left.\frac{d^2 A}{dt^2}\right|_{t=0} = \int_D (|\nabla h|^2 + 2Kh^2)\, dA, \tag{2.1}$$

where h is as in the previous paragraph.

Clearly the stability of D is equivalent to the inequality:

$$\int_D |\nabla h|^2 dA > -2 \int_D K h^2 dA$$

for any h in $C^\infty(\overline{D})$ which vanishes on Γ. Recall now that for a minimal surface, the pullback metric of the metric on S^2 under the Gauss map is given by:

$$d\hat{s} = -K ds^2, \tag{2.2}$$

from which it follows that

$$d\hat{A} = -K dA$$

and

$$|\nabla h|^2 = -K|\hat{\nabla} h|^2,$$

where $\hat{\nabla}$ is the gradient of the new (pullback) metric on Σ. Hence we can rewrite the previous equation in the form:

$$\int_D |\hat{\nabla} h|^2 \, d\hat{A} > 2 \int_D h^2 \, d\hat{A}.$$

Now recall that the ratio

$$Q_D(h) = \frac{\int_D |\nabla h|^2 \, dA}{\int_D h^2 \, dA}$$

is the *Rayleigh quotient*, and the quantity

$$\lambda_1(D) = \inf Q_D(h)$$

represents the first (smallest) eigenvalue of the Dirichlet problem

$$\begin{cases} \Delta h + \lambda h & = 0 \quad \text{in D} \\ h & = 0 \quad \text{on } \Gamma. \end{cases}$$

The above infimum can be taken over all continuous piecewise smooth functions on the closure of D which vanish on Γ, and Δ represents the Laplace operator with respect to the (given) metric on D. If Γ is sufficiently smooth, then the above boundary value problem has a solution h_1 corresponding to the eigenvalue λ_1, and the above infimum is attained by $h = h_1$. Putting these facts together, we see that the above conditions of stability are equivalent to the inequality:

$$\lambda_1(D) > 2.$$

For a minimal surface, the metric (2.3) is exactly the pullback metric under the Gauss map of the metric on S^2. Thus we have the following theorem, due to Schwarz:

Theorem 2.4 (Schwarz) *Let D be a (relatively) compact domain on a minimal surface $\Sigma \hookrightarrow \mathbb{R}^3$. Suppose that the Gauss map g takes D injectively on a domain $\hat{D}$ on the unit sphere S^2. If $\lambda_1(\hat{D}) < 2$, then D cannot be locally area-minimizing with respect to its boundary.*

Proof. Because of the above observations we see that the condition $\lambda_1(\hat{D}) < 2$ implies the existence of a C^∞ function $u : \overline{D} \to \mathbb{R}$ such that $u_{|\partial\hat{D}} = 0$ and which makes the second variation of area (2.2) negative. Namely D is the initial point of a 1-parameter family of minimal surfaces with boundary equal to ∂D, such that each element in this family has area smaller than the area of D. Hence D cannot be locally area-minimizing (and therefore cannot be stable). q.e.d.

Schwarz's theorem, and the facts that the first eigenvalue of the Laplacian on a hemisphere of S^2 is equal to 2, and that $D' \subseteq D \Rightarrow \lambda_1(D') \geq \lambda_1(D)$, imply that if the Gauss map g is injective on a domain D of a minimal surface, and the Gaussian image $g(D)$ contains a hemisphere on S^2, then D is unstable. Barbosa and Do Carmo [3, 4] generalized this result, showing that g does not necessarily have to be injective, by proving the following theorem.

Theorem 2.5 (Barbosa-Do Carmo) *If the area of the Gaussian image $g(D) \subseteq S^2$ is less than 2π, then D is stable.*

Remark 2.1 *Note that the converse statement to theorem* 2.5 *does not hold. This can be seen by considering two or more disjoint stable minimal annuli, and by gluing them together by thin bridges, to obtain a new connected stable minimal surface (see [21], [**?**], [38], [39]). This new minimal surface is stable, but in general its image under the Gauss map, counted with multiplicities, has area larger that 2π.*

We recall yet another characterization [3, 4] of stable minimal surfaces. First we need the following definition.

Definition 2.1 *A smooth normal vector field $V(p) = p + u(p)\vec{N}(p)$, where $\vec{N}(p)$ is the normal vector to D in p is said to be a Jacobi vector field if $u : \overline{D} \to \mathbb{R}$, $p \mapsto u(p)$ satisfies the Jacobi equation*

$$-\Delta u + 2uK = 0,$$

where Δ is the Laplace operator on D and K is the Gaussian curvature on D.

Then it can be shown, via the Morse Index Theorem [35], that

Theorem 2.6 (Classical) *A domain D is stable if and only if no subdomain D' inside D admits a Jacobi vector field which is nonzero in D' but zero on $\partial D'$.*

2.3 The maximum principle

Suppose that $X = (X_1, X_2, X_3) : S \to \mathbb{R}^3$ is a minimal immersion. Then one can check, using the definition of zero mean curvature (since $\Delta X = 2H \cdot N$), that the coordinate functions X_1, X_2, X_3 are harmonic functions. It is well known that there is a maximum principle for harmonic functions. Hence there is also a maximum principle for minimal surfaces.

Theorem 2.7 (Maximum principle at an interior point) *Let M_1 and M_2 be two minimal surfaces in $\mathbb{R}^3$ that intersect at an interior point p. If M_1 lies on one side of M_2 near p, then M_1 and M_2 coincide in a neighborhood of the point p.*

Theorem 2.8 (Maximum principle at a boundary point) *Suppose that the minimal surfaces M_1 and M_2 have boundary curves C_1 and C_2 respectively. Let p be a point in $C_1 \cap C_2$, and suppose that $T_p(M_1) = T_p(M_2)$ and $T_p(C_1) = T_p(C_2)$. Choose orientations of M_1 and M_2 in such a way that the two surfaces have the same normal vector $\vec{n}$ at p. If near p M_1 lies on one side of M_2 and the conormal vectors of M_1 and M_2 at p coincide, then M_1 and M_2 coincide in a neighborhood of the point p.*

2.4 Minimax theorems

It is intuitively evident that a smooth function f from $\mathbb{R}^N$ to $\mathbb{R}$ which has two isolated nondegenerate relative minimum points should also have at least one unstable critical point. This was proven by Courant in his book [6].

Later in this paper we will need the existence of a *minimax solution* between two stable minimal surfaces with the same boundaries (see the Introduction and section 4).

Definition 2.2 *Let M be the space of embedded surfaces contained in the product region of space bounded by $\Sigma_1 \cup \Sigma_2$, and having interior angles less than π. Let $\overline{M}$ be an opportunely constructed compactification of M (see [11, p. 234-235]), and let $\mathcal{F}$ be a cover of $\overline{M}$ by connected sets. Denote with F a generic element in $\mathcal{F}$. Define now the minimax of the function $A =$ "Area on $\mathcal{F}$" by:*

$$Minimax(A, \mathcal{F}) = inf_{F \in \mathcal{F}} sup\{A(M) | M \in F\}.$$

Then the deep minimax theorems proven by Pitts and Rubinstein [30], and generalized by Jost [11] to the case of nonempty boundary, guarantee that the following assertion holds true.

Theorem 2.9 (Minimax) *There exists an embedded unstable minimal surface* Σ^**, contained in the region of space* $\mathcal{R}$ *bounded by* $\Sigma_1 \cup \Sigma_2$*, having* $\partial(\Sigma^*) = \partial(\Sigma_1) = \partial(\Sigma_2)$*. Furthermore, the genus of* Σ^* *is at most equal to the genus of* Σ_i*,* $i = 1, 2$*.*

Remark 2.2 *In our case, it will be very important to make sure that* Σ^* *is homeomorphic to each of* Σ_1 *and* Σ_2*. More precisely, the minimax theorems mentioned above guarantee that the genus of* Σ^* *is at most equal to* g*, but it could,* a priori*, be less than* g*. However, since* Σ^* *is contained in the product region* $\mathcal{R}$ *bounded by* $\Sigma_1 \cup \Sigma_2$*, and both* Σ_1 *and* Σ_2 *are incompressible in* $\mathcal{R}$*, then the genus of* Σ^* *is at least* g*. Hence the unstable minimax solution is homeomorphic to the two stable ones.*

2.5 Nitsche's 4π-theorem

Finally, let us state here another result which will be useful in subsequent sections of this paper, namely the "4π-theorem", due to J.C.C. Nitsche [25].

Theorem 2.10 (Nitsche) *Let* γ *be a real analytic curve in* $\mathbb{R}^3$ *having total curvature less than* 4π*. Then* γ *bounds a unique minimal surface which is topologically equivalent to a disk, and this disk is stable and has no branch points.*

Sketch of Proof. It follows easily from previous work of Shiffman that if γ bounded more than one minimal disk which is a C^0-minimum to area, then it would also bound a branched minimal disk that is not a C^0-minimum to area. Nitsche proved, for the curve γ stated in the theorem, that every minimal disk which it bounds is a C^0-minimum for area, and hence by Shiffman's result it must be unique. q.e.d.

3.0 Existence and description

In this section we show the existence of the minimal surfaces $\Sigma(V, t)$, and describe them geometrically, for sufficiently small values of t.

3.1 Existence

Recall that P_t denotes the plane $\{(x,y,z)\in\mathbb{R}^3|z=t\}$. Let us consider two closed smooth Jordan curves, α and $\beta=\beta(0)$ in the plane $P_0 : z=0$ of $\mathbb{R}^3$. Suppose that the curves α and β intersect transversely in a finite number of points $p_1,\ p_2,\ldots,\ p_N$, which we will call *crossing points*.

For our purposes in this paper, the word *varifold* will denote a finite union of connected compact planar regions which are smooth except possibly at a finite number of boundary points, and to the interior of each such component we associate a nonnegative integer, called the *multiplicity* of that component, without orientation. We will denote by $\beta(t)$ the translated image of β by the vector $(0,0,t)$.

First of all we will prove an existence theorem, which shows that the class of minimal surfaces that are our object of study is not the empty one, and that it makes sense to study the asymptotic behavior.

Theorem 3.1 *Let α and β be two Jordan curves in the plane P_0. Let V be a fixed varifold in P_0 with $\mathbb{Z}_2$-boundary given by $Z=\alpha\cup\beta$. Then, for every t sufficiently close to 0, there exists a compact stable embedded minimal surface $\Sigma(t)$ whose boundary is $\alpha\cup\beta(t)$, and such that $\Sigma(t)$ is "close to" V, in the sense that the sequence of minimal surfaces $\{\Sigma(t)\}_t$ converges to V, as $t\to0$.*

The proof will follow from the following theorem proven by Meeks and Yau [21], which we state here without proof.

Theorem 3.2 (Meeks-Yau) *Let M be a submanifold of a 3-dimensional analytic manifold whose boundary is piecewise smooth and has nonnegative mean curvature with respect to the inner pointing normal vector, and has interior angles less than π. Let Σ be a compact subdomain of ∂M such that Σ is incompressible in M, namely each homotopically nontrivial curve in Σ is also homotopically nontrivial in M. Then there exists a stable minimal embedding $f:\Sigma\to M$ such that $f(\partial\Sigma)=\partial\Sigma$. Moreover if $g:\Sigma'\to M$ is another minimal immersion of a compact surface such that $g(\partial\Sigma')=\partial\Sigma$, then one can assume $f(\Sigma)\subset M\backslash g(\Sigma')$.*

Proof of Theorem 3.1. Let $\mathcal{R}(\alpha)$ and $\mathcal{R}(\beta)$ be the bounded simply connected components of $P_0\backslash\alpha$ and $P_0\backslash\beta$ respectively, and let $\mathcal{R}$ be the region in the plane P_0 given by $\mathcal{R}=\mathcal{R}(\alpha)\cup\mathcal{R}(\beta)$.

We wish to construct the analytic three-manifold $M(V,t)$ satisfying the hypotheses of theorem 3.2, for t sufficiently small. Let $C_1,\ldots,C_k$ be the

bounded components of $P_0\backslash Z$ with V-multiplicity equal to zero. Choose round circles $S_i \subseteq Int(C_i)$, for each $i\in\{1,\dots,k\}$, and let $S_i(t)$ be the vertical upward translated of S_i by t. Since the number of connected components of $P_0\backslash(\alpha\cup\beta)$ is finite, there will be a largest t' such that for $t\leq t'$ there exist stable catenoids $A_i(t)$ bounded by $S_i\cup S_i(t)$ whose height is t. We do not place catenoids above any component of V which has multiplicity 2, nor above components of V whose multiplicity is 1.

Next, glue flat components away from the $(0,1,0,1)$ crossing points, and where the prescribed multiplicities are 1 or 2. The glueing above the multiplicity 2 components should be made so that the following condition is satisfied.

Let σ be the multiplicity two arc bounded by the two $(0,1,2,0)$ crossing points. Then of the two flat pieces glued above the multiplicity 2 component, one does not contain σ, and it glues on smoothly (as its projection "crosses" σ) to the adjacent flat piece which also doesn't contain σ and projects on the multiplicity 1 component "across" σ.

Finally, along the $(0,1,0,1)$ crossing points, glue the corresponding multiplicity 1 pieces via a twisted strip (the glueing taking place along the "helices"), in such a way that if $\mathcal{S}$ is the surface just constructed, then the curvature of the strip with respect to the inner pointing normal vector (with respect to $\mathcal{S}$) is nonnegative.

Then the three-manifold $M = M(V,t)$ is given by

$$M = Slab(t)\backslash(\bigcup_{i=1}^{k} Int(A_i(t))\bigcup(Out(\mathcal{S}))),$$

where $Slab(t)$ is the region of space between the planes P_0 and P_t, $Int(A_i(t))$ is the bounded simply connected region in $Slab(t)$ containing $A_i(t)$ in its boundary, $Out(\mathcal{S})$ is the unbounded connected component of $Slab(t)\backslash\mathcal{S}$. Note that M is analytic because the boundaries of the catenoids are circles. Also, by construction, the interior angles are less than π, and the curvature with respect to the inner pointing normal is nonnegative.

Then the surface Σ satisfying the hypotheses of theorem 3.2 is going to be $\mathcal{S}$.

By construction, the surface Σ constructed in this way has

$$\pi_1(\Sigma) = \mathcal{F}(k),$$

where $\mathcal{F}(k)$ denotes the free group on k generators. Moreover Σ admits a submersion onto a planar region in the plane P_0 which is homotopically equivalent to $V\subseteq\mathcal{R}$. Now, V is easily seen to be homotopically equivalent

to M (which is, homotopically, the cross product of the unit interval with a plane from which k disks have been removed), and so one has

$$\pi_1(M) = \mathcal{F}(k).$$

This allows us to conclude that Σ is incompressible in M because of the commutativity of the following diagram:

$$\begin{tikzcd}
\pi_1(\Sigma) \arrow[rr] \arrow[dr, "1\text{-}1"'] & {} \arrow[d, phantom, "\circlearrowright"] & \pi_1(M) \arrow[dl, leftrightarrow, "\textit{isomorphism}"] \\
& \pi_1(V) &
\end{tikzcd}$$

Hence the above construction satisfies all the hypotheses of Meeks and Yau's theorem, providing us with a *good barrier for the solution of Plateau's problem*. We are assured the existence of a compact stable embedded minimal surfaces $\Sigma(t)$, having boundary given by $\alpha \cup \beta(t)$, for all $t \le t'$, and incompressible in M. By construction, we have that $lim_{t\to 0}(\Sigma(t)) = V$.

q.e.d.

Remark 3.1 *As the number of annuli used as barriers in the previous proof increases, the genus of the minimal surfaces $\Sigma(V,t)$ correspondingly obtained increases, and the area of $\Sigma(V,t)$ decreases, since the area of V decreases. Hence the least area surfaces bounded by $\alpha\cup\beta(t)$ are the ones having largest genus, and they correspond to the only varifold which has no components with multiplicity* 2*. The surfaces of largest area bounded by $\alpha\cup\beta(t)$ correspond to the disjoint union of the regions $\mathcal{R}(\alpha)$ and $\mathcal{R}(\beta(t))$.*

3.2 Description

In this section we will describe geometrically the minimal surfaces whose existence has been shown in the previous section, when the distance between the two planes containing the boundary of such surfaces is sufficiently small.

Let's give here a definition which will be useful in the proof of the next theorem.

Definition 3.1 *A sequence of surfaces $\{S_i\}_{i=1}^{\infty}$ with boundary is said to converge to a proper surface S in $\mathbb{R}^3$ if, for each compact region B of $\mathbb{R}^3$, there exists a positive integer N_B such that, for $i > N_B$, $S_i \cap B$ is a*

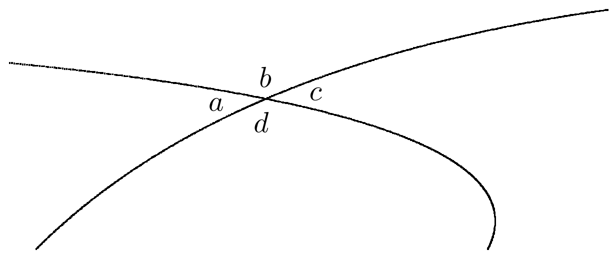

Figure 3.1: The varifold V, with multiplicity, around a crossing point.

normal graph over S, and $\{S_i \cap B\}_{i=1}^{\infty}$ converges to $S \cap B$ in the C^1 norm ($\|f\|_{C^1} = \|f\|_0 + \|Df\|_0$). Moreover, if $\{S_i\}_{i=1}^{\infty}$ is a sequence of surfaces with boundary, and $\{\phi_i\}_{i=1}^{\infty}$ is a sequence of homotheties of $\mathbb{R}^3$ with center the origin, and $\{\phi_i(S_i)\}_{i=1}^{\infty}$ converges to a connected piece of a helicoid bounded by two straight lines, we will say that S_i is approximately helicoidal near the origin for i sufficiently large. Similarly one can define the convergence of a continuous family $\{S_t\}$, as t approaches t_0.

The minimal surfaces considered in the Main Theorem were partially described by Wayne Rossman [31], under the hypotheses that these minimal surfaces exist and are actually area-minimizing. By theorem 3.1 we have proven the existence of stable embedded minimal surfaces which are not necessarily area-minimizing (see the above remark). In this section we prove theorem 3.7, which will provide a complete description of the *stable* minimal surfaces under consideration in this paper.

Before stating the theorem, let us make a few observations about the crossing points. More precisely, consider a varifold V, arising as the limit of a family $\Sigma(t_i)$ of minimal surfaces, in a small neighborhood of the crossing points. This is pictured schematically in Figure 3.1, where a, b, c, d represent the multiplicity of V in each of the four regions. Then the following claim holds.

Claim 3.1 *Let V be a varifold determined by any of the minimal surfaces obtained in theorem* 3.1*. Then, with notation as introduced above, the following three properties hold for V:*

- *The difference between the multiplicity of any region and an adjacent one in Figure* 3.1 *is exactly one.*
- *At least one of a, b, c, d must be equal to zero.*

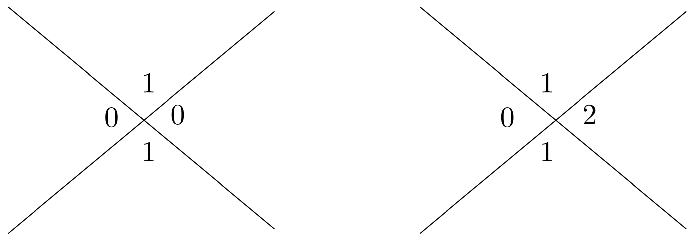

Figure 3.2: Crossing points with multiplicity $(0, 1, 0, 1)$ and $(0, 1, 2, 1)$.

- *The only possibilities allowed for the multiplicity at the crossing points are* $(0, 1, 0, 1)$ *and* $(0, 1, 2, 1)$.

Proof. The proof of the three assertions in the claim follows easily from the fact that the surfaces being considered are embedded and stable, and that Z is the homology boundary of these surfaces. q.e.d.

Hence we will, from now on, speak of points with multiplicity $(0, 1, 0, 1)$ and $(0, 1, 2, 1)$. The two cases are illustrated in Figure 3.2.

Remark 3.2 *The previous claim implies that, for t sufficiently small, the varifold associated with each embedded compact stable minimal surface bounded by $\alpha \cup \beta(t)$ is actually in the set $\mathcal{V}(\alpha, \beta)$, whose meaning is the same as in the introduction.*

Theorem 3.3 *Let $\Sigma(t)$ be a compact stable embedded minimal surface with boundary given by the union of the curves α and $\beta(t)$. Then, for t sufficiently close to 0, the surface $\Sigma(t)$ has the following properties:*

1. *In the complement of the union of small cylindrical neighborhoods $N(i)$ of the crossing points, the connected components of $\Sigma(t) \backslash \cup N(i)$ are graphs over their projection to the plane P_0.*

2. *In $N(i)$, $\Sigma(t)$ is either approximately helicoidal, or it is the union of two graphs over the plane P_0.*

Proof. First notice that the theorem certainly holds if $\Sigma(t)$ consists of the union of the two planar regions bounded by α and $\beta(t)$. Hence we need to prove the theorem for connected $\Sigma(t)$. The proof will be divided in five steps.

Step 1: *For t sufficiently close to 0, the components of $\Sigma(t)$ away from the crossing points are graphs over P_0.*

For each $1\leq i\leq n$, let $D_i(\epsilon)$ be a disk in P_0 with center p_i and radius ϵ, chosen so that the following two conditions are satisfied:

(a) the disks D_i are mutually disjoint;

(b) for each i, $D_i\cap(\alpha\cup\beta)$ is topologically a $\times$.

Let $N = P_0\backslash\bigcup_{i=1}^{N} D_i(\epsilon)$, and let $U = N\times\mathbb{R}$ be the vertical cylinder over N. Moreover, let $\hat{\Sigma}(t) = \Sigma(t)\cap U$. Let us notice that along the curves $\alpha\cap\hat{\Sigma}(t)$ and $\beta(t)\cap\hat{\Sigma}(t)$, the normal vector to $\Sigma(t)$ must become arbitrarily vertical as t approaches 0. To see this, for example along $\beta(t)\cap\hat{\Sigma}(t)$, suppose $\Sigma(t)$ is vertical at a boundary point p of $\beta(t)\cap\hat{\Sigma}(t)$. Consider, above and away from $\Sigma(t)$, a piece of a half strictly unstable vertical catenoid with boundary curves a larger circle in the plane P_0 and a smaller circle in the plane $P_{t+\epsilon'}$ (this catenoid is a graph above the plane P_0, except along the smaller boundary circle). Translate vertically down this catenoid piece until it makes first contact with $\Sigma(t)$ at some point p; the maximum principle implies first that p is a boundary point both of the catenoid piece and of $\Sigma(t)$, and then that the two minimal surfaces $\Sigma(t)$ and the catenoid piece coincide in a small neighborhood of p. This contradiction shows that $\Sigma(t)$ cannot become vertical along its boundary, away from the crossing points. Moreover we have that locally the surface $\hat{\Sigma}(t)$ is situated on the same side with respect to the vertical cylinder $((\alpha\cup\beta)\cap\hat{\Sigma}(t))\times\mathbb{R}$, otherwise one could translate a catenoid in such a way that the first point of contact would be an interior point, which contradicts the maximum principle. With the same notation as above, let us notice that there exists the upper lower bound of the angle $\theta(p,t)$ formed by $\Sigma(t)$ with an arbitrary catenoid piece intersecting $\Sigma(t)$ only in p; in fact this infimum is given by the angle between the tangent plane to $\Sigma(t)$ at p and the horizontal plane P_0. Moreover, for an arbitrary p in $(\alpha\cup\beta(t))\cap\hat{\Sigma}(t)$, the upper lower bound of $\theta(p,t)$ approaches zero as t gets closer to 0. Let

$$\theta_0(t) = \max\{\theta(p,t)|\, p\in(\alpha\cup\beta(t))\cap\hat{\Sigma}(t)\}.$$

The compactness of $(\alpha\cup\beta(t))\cap\hat{\Sigma}(t)$ guarantees that $\theta_0(t)$ is well defined. Furthermore, what has been said above implies that

$$\lim_{t\to 0}\theta_0(t) = 0, \quad \text{and therefore} \quad \lim_{t\to 0}\tan\theta_0(t) = 0.$$

By R. Schoen's estimate [33] there exists a universal constant c such that $K < \frac{c}{r^2}$, where K is the Gaussian curvature at a point of a stable minimal surface and r is the distance between that point and the boundary of the surface. Now let us choose t sufficiently close to 0 so that $\tan\theta_0(t) < \min\{\frac{\pi}{32\sqrt{c}}, \frac{1}{16}\}$. Let p be a point in $\hat{\Sigma}(t)$, and let $\hat{p}$ be the orthogonal projection of p on the plane P_0. Let r be the horizontal distance between $\hat{p}$ and $\alpha\cup\beta\backslash\bigcup_{i=1}^{n} D_i$. Let Cyl_1 be the part of the vertical cylinder $D(\hat{p}, \frac{r}{2})\times\mathbb{R}$ with height $2r\tan\theta_0$ and containing $\hat{\Sigma}(t)\cap\partial Cyl_1$ in its interior. Let Cyl_2 be the part of the vertical cylinder $D(\hat{p}, \frac{r}{4})\times\mathbb{R}$ with height $2r\tan\theta_0$ which is contained in Cyl_1. Let us suppose that our assertion in Step 1 is not true. It will be therefore possible to find a point q in $\Sigma(t)\cap Cyl_2$ whose normal vector $\vec{N}(q)$ is horizontal, namely $< \vec{N}(q), \vec{e_3} >= 0$. There follows the existence of a geodesic $\omega(s)$ in $\Sigma(t)$ with unimodular velocity, and such that $\omega(0) = q$, $\omega'(0) = \vec{e}_3$, and the curvature $|\omega''(s)|$ of $\omega(s)$ in Cyl_1 is bounded above by $\frac{2\sqrt{c}}{r}$ because of Schoen's estimate.

Notice that, since $\tan\theta_0 < \frac{1}{16}$ (and since $|\omega'(s)| = 1$ implies $|d\omega|\approx|ds|$), $\omega(s)$ lies in $D(\hat{p}, \frac{r}{2})\times\mathbb{R}$ for $0\leq s\leq 4r\tan\theta_0$. Considering the estimate for the curvature of $\omega(s)$, the estimate $\frac{\pi}{32\sqrt{c}}$ for $\tan(\theta_0)$, and the integral

$$\int_0^{4r\tan\theta_0} |\omega''(s)|\, ds,$$

we can conclude that the length of the curve $\omega'(s)$ in the unit sphere S^2 is less than $\frac{\pi}{4}$, for $0\leq s\leq 4r\tan\theta_0$. Since $\omega'(0) = \vec{e}_3$, we have that $< \omega'(s), \vec{e}_1 >$ and $< \omega'(s), \vec{e}_2 >$ are both less than $\frac{\sqrt{2}}{2}$, and that $< \omega'(s), \vec{e}_3 >$ is larger than $\frac{\sqrt{2}}{2}$, for $0\leq s\leq 4r\tan\theta_0$. Hence it is

$$\langle\omega(4r\tan\theta_0) - \omega(0), \vec{e_3}\rangle = \int_0^{4r\tan\theta_0} \langle\omega'(s), \vec{e}_3\rangle ds > \frac{\sqrt{2}}{2} 4r\tan\theta_0 > 2r\tan\theta_0.$$

Likewise we have:

$$|\langle\omega(4r\tan\theta_0) - \omega(0), \vec{e}_1\rangle| < \frac{r\sqrt{2}}{8},$$

and

$$|\langle\omega(4r\tan\theta_0) - \omega(0), \vec{e}_2\rangle| < \frac{r\sqrt{2}}{8},$$

because of the condition $\tan\theta_0 < \frac{1}{16}$. But this implies that $\omega(s)$ intersects one of the two horizontal disks of ∂Cyl_1 for some value of s between 0 and

$4r\tan\theta_0$, which is impossible, given the way Cyl_1 was constructed. Hence $\vec{N}$ is never horizontal on $\hat{\Sigma}(t)$, which therefore is union of graphs over $D(\hat{p}, \frac{r}{4})$.

Moreover, notice that at most two sheets can lie over $D(\hat{p}, \frac{r}{4})$, otherwise $\Sigma(t)$ would not be embedded (at least along the boundary). If the surface is area-minimizing, then at most one sheet lies above each point, but if no assumption of area minimality is made, then there could be points above which there are two sheets.

Let us remark that for $t \approx 0$ the above graphs are "almost horizontal", since it is clear the above argument can be strengthened to show that, for any $x \in (0,1)$, it is possible to make $|\langle \vec{N}(q), \vec{e_3} \rangle| > x$ for each $q \in \Sigma(t)$, for t sufficiently close to 0.

Step 2: $\Sigma(t)$ *is the union of two disjoint graphs in the neighborhood of each* p *in* $\partial V_1 \cap \partial V_2$*, where* V_1 *and* V_2 *are components of* V *with multiplicities* 1 *and* 2 *respectively.*

To see this, consider a small vertical cylinder $\mathcal{C}$ whose vertical axis contains p and whose height is t. Moreover, let $\hat{\Sigma}(t) = \Sigma(t) \cap \mathcal{C}$. Now homotetically expand $\mathcal{C}$, with coefficient of homothety $1/t$, and center of the homothety in p. The image of $\hat{\Sigma}(t)$ under the homothety converges, as $t \to \infty$, to a minimal surface having a simply connected component S which is a stable minimal surface contained in a half-space and bounded by a straight line, ℓ. The image $g(\ell)$ of ℓ via the Gauss map is either a single point or a great-circle on S^2 containing the north and south pole of S^2, and near the boundary ℓ the image of S via the Gauss map is entirely contained in one of the two hemispheres determined by $g(\ell)$, because of the way the barrier to get $\Sigma(t)$ was constructed (theorem 3.1). Hence the image under the Gauss map of S is entirely contained in such half hemisphere, by the hypothesis of stability. By reflection with respect to the line ℓ one obtains a complete minimal surface containing a line and having total curvature between -4π and 0. The two only complete minimal surfaces with total curvature -4π are the catenoid and Enneper surface [2, p.40]. Since the catenoid does not contain a line of reflective symmetry, and Enneper surface's Gauss map does not satisfy our conditions, then S must be a half plane, and its image via the Gauss map must ba a point. Finally, notice that the points contained in the other simply connected component of the homothetic expansion of $\hat{\Sigma}(t)$ correspond to points which are contained in the interior of $\Sigma(t)$, and hence Schoen's curvature estimate applies to them. These observations complete the proof.

Step 3: $\Sigma(t)$ *is the union of two disjoint graphs in the neighborhood of each crossing point having multiplicity* $(0, 1, 2, 1)$.

This follows from the previous two steps, and from the analysis of the

possible liftings to $\Sigma(t)$ of small circles in V around the point under consideration.

Our next aim is to study $\Sigma(t)$ around a crossing point whose multiplicity is $(0,1,0,1)$.

Step 4: $\Sigma(t)$ *is topologically a disk in the neighborhood of each crossing point with multiplicity* $(0,1,0,1)$.

We will show that in a closed cylindrical neighborhood U of each crossing point with multiplicity $(0,1,0,1)$, the compact surface $\Sigma(t)\cap U$ has genus zero for t sufficiently close to 0. In order to do this, let us homothetically expand the spherical neighborhood U with center in a point p, with expansion coefficient $\frac{1}{t}$, where p is a point of maximum Gaussian curvature inside $\Sigma(t)\cap U$. Let us denote by $\tilde{U}$ the expanded neighborhood, and notice that the expansion transforms the planes P_0 and P_t to two new planes which have distance equal to 1 from each other. Moreover the homothety takes the arcs $\alpha\cap U$ and $\beta(t)\cap U$ to arcs $\tilde{\alpha}$ and $\tilde{\beta}(t)$ in $\tilde{U}$ which are segments of an almost straight line. Let $\partial\tilde{\alpha} = \{a_1, a_2\}$, and $\partial\tilde{\beta}(t) = \{b_1, b_2\}$, with the convention that the orthogonal projections of a_1 and b_1 on P_0 lie in the boundary of the same multiplicity one component of the interior of V, and the orthogonal projections of a_2 and b_2 lie in a different multiplicity one component. Now let us join a_1 to b_1 and a_2 to b_2 by two geodesic arcs contained in the homothetic expansion $\tilde{\Sigma}(t)$ of $\Sigma(t)$. Because $\Sigma(t)$ is a graph over P_0 away from the crossing points with multiplicity $(0,1,0,1)$, and because for each $x\in(0,1)$ we have $|\langle\vec{N}(q),\vec{e}_3\rangle| > x$ away from these crossing points, for $t\approx 0$ (namely the normal vector to $\Sigma(t)$ in q is almost vertical if q is away from the crossing points with multiplicity $(0,1,0,1)$), we may assume that the two geodesic arcs defined above project orthogonally on two different multiplicity one components of the interior of V, and hence do not intersect each other. We will now show that the piece $\bar{\Sigma}(t)$ contained in $\tilde{\Sigma}(t)$ and bounded by the loop union of the four curves $\tilde{\alpha}$, $\tilde{\beta}(t)$, and the two geodesic arcs previously defined, is a disk. Since $\partial(\Sigma(t))$ is contained inside the boundary of the convex hull of $\Sigma(t)$, $\Sigma(t)$ separates its convex hull (which is simply connected) into two distinct types of regions, one associated with the "+" sign and the other with the "−" sign. Hence $\Sigma(t)$ is orientable, and consequently also $\bar{\Sigma}(t)$ is so. This implies that the Gauss map, from the oriented $\bar{\Sigma}(t)$ to the unit sphere S^2, is well defined. The Gaussian image $\nu(t)$ of the boundary curve $\partial\bar{\Sigma}(t)$ is a curve that lies in a small neighborhood of the spherical region bounded by the union of two great semicircles joining the north and south poles of S^2. Since $\Sigma(t)$ is stable, $\bar{\Sigma}(t)$ is also stable. Because the Gaussian image of a stable minimal surface cannot contain a

hemisphere, the image of $\bar{\Sigma}(t)$ under the Gauss map can only contain one of the two regions in the complement of this neighborhood in S^2, and must be disjoint from the other region. Since the winding number of the Gauss map around $\partial\bar{\Sigma}(t)$ is one, the Gauss map can only cover this region once. This implies that for t sufficiently close to 0 it is:

$$-2\pi \leq \int_{\bar{\Sigma}(t)} K\, dA < 0.$$

Now, the geodesic curvature k_g is zero along the two almost straight arcs contained in $\partial\bar{\Sigma}(t)$, and along $\tilde{\alpha}$ and $\tilde{\beta}(t)$ the geodesic curvature is approximately equal to zero. The sum of the exterior angles where these smooth arcs intersect is between 0 and 4π. Therefore by the Gauss-Bonnet theorem, one has that the Euler characteristic of $\bar{\Sigma}(t)$ is either zero or one. Since $\partial\bar{\Sigma}(t)$ consists exactly of one curve, we can conclude that the Euler characteristic is 1, and that $\bar{\Sigma}(t)$ is topologically a disk. Hence, in small neighborhoods of the crossing points with multiplicity $(0,1,0,1)$, $\Sigma(t)$ is topologically a disk.

Step 5: $\Sigma(t)$ *is approximately helicoidal around crossing points having multiplicity* $(0,1,0,1)$.

Normalize $\Sigma(t)\cap N$ by a homothety with center a point p of maximum Gaussian curvature, in such a way that $\max_{q\in\Sigma(t)\cap N}\{|K(q)|\} = 1$ on the normalized surface, which we shall denote by $\breve{\Sigma}(t)$. Modulo a translation, we can suppose that the point $p\in\beta$ is the origin. Let us notice that $\max_{q\in\Sigma(t)\cap N}\{|K(q)|\} \to \infty$ on $\Sigma(t)\cap N$ as t approaches 0, since the Gauss map ν is almost vertical on $\partial\Sigma(t)$, except near the crossing points where ν changes very quickly, and hence the modulus of $K = Jac(d\nu)$ must be large near the crossing points. Therefore normalizing $\max_{q\in\Sigma(t)\cap N}\{|K(q)|\} = 1$ involves a dilation factor which becomes arbitrarily large as t approaches 0, and hence the two planar curves in the boundary of $\breve{\Sigma}(t)$ become arbitrarily straight as t approaches 0, around the crossing points. A result of Anderson [1] states that for each sequence of surfaces $\{\breve{\Sigma}_{t_\ell}\}_{\ell=1}^{\infty}$, $t\to\infty$, it is possible to extract a convergent subsequence (in the C^2 norm) $\{S_{ij}\}_{i=1}^{\infty}$ in the compact spherical neighborhood $B(0,j)$ with radius j in $\mathbb{R}^3$. This sequence of surfaces can be chosen in such a way that $\{S_{ij}\}_{i=1}^{\infty}$ is a subsequence of $\{S_{kl}\}_{k=1}^{\infty}$ if $j > l$. The sequence $\{S_{mm}\}_{m=1}^{\infty}$ is the Cantor diagonalization, and converges in the C^2 norm in arbitrary compact regions to a surface S having one or two boundary curves, which must be straight lines. Moreover Anderson's compactness theorem [1] implies that S is embedded. The limit surface S is simply connected in any compact spherical neighborhood, and therefore is simply connected in $\mathbb{R}^3$. If the boundary of S consists of two straight lines, then by Alexandrov reflection principle

S can be extended to a simply connected minimal surface in $\mathbb{R}^3$ properly embedded, without boundary and with infinite symmetry group. By virtue of a theorem of Meeks and Rosenberg [17], such extended minimal surface is a plane or a helicoid. However this surface contains two straight lines which do not intersect and are not parallel to each other, and hence S is a piece of a helicoid, with one or two boundary lines. If S had only one boundary line, then it could be extended via a rotation of angle π around the straight boundary line, producing a properly embedded minimal surface. Because the convergence of the above subsequence to S is with respect to the C^2 norm, the normal vectors are converging as well, and the extended surface has finite total curvature. By a result of López and Ros such a surface must be a plane or a catenoid; however since it is simply connected, it must be a plane. Hence S is a half-space. Since the convergence to S is of class C^2, we know that the normal vectors along ∂S are not constant as $m \to \infty$, which implies that S cannot be a plane. Therefore S must be a piece of helicoid with two boundary lines. Since $\hat{\Sigma}(t)$ is a graph over P_0 and because we are considering a point with multiplicity $(0,1,0,1)$, $\hat{\Sigma}(t)$ is a graph over the components of $P_0 \backslash (\beta(0) \cup \alpha)$ having the "+" sign. Hence S is totally determined. If there was some subsequence which would not be eventually contained in a given ϵ-neighborhood of S in a given sphere $B(0,j)$ in $\mathbb{R}^3$, then it would be possible to find a subsequence $\{\check{\Sigma}_{t_\ell}\}_{\ell=1}^{\infty}$ converging to a point not belonging to S, which is a contradiction. This allows us to conclude that any sequence $\{\check{\Sigma}_{t_\ell}\}_{\ell=1}^{\infty}$ such that $t_\ell \to 1$ eventually lies in a predetermined arbitrarily small ϵ-neighborhood of S inside each compact region of $\mathbb{R}^3$. For an arbitrarily given ϵ, let us choose ℓ big enough in such a way that the surface $\check{\Sigma}_{t_\ell} \cap D_j(0)$ be contained in an ϵ-neighborhood of $S \cap D_j(0)$. For each pair of points p and q, with p in $\check{\Sigma}_{t_\ell}$ with normal vector $\vec{N}(p)$ to $\check{\Sigma}_{t_\ell}$ and q in S with normal vector $\vec{N}(q)$ to S such that $dist(p,q) < \epsilon$, the estimate on the function $|K(q)|$ implies that $| < \vec{N}(p), \vec{N}(q) > |$ is bounded away from zero. In fact by choosing ϵ sufficiently close to zero and ℓ sufficiently large, we will be able to achieve $| < \vec{N}(p), \vec{N}(q) > |$ to be arbitrarily close to 1. It follows that $\check{\Sigma}_{t_\ell}$ is union of graphs on S for ℓ sufficiently large, and hence $\check{\Sigma}_{t_\ell}$ is a one-sheeted graph on S around a crossing point with multiplicity $(1,0,1,0)$, which is in accordance to the end of the proof of Part 1. This allows us to conclude that $\{\check{\Sigma}_{t_\ell}\}_{\ell=1}^{\infty}$ converges to S in the C^0-norm as one-sheeted graphs, and that the normal vectors are convergent as well; hence we have in addition that $\{\check{\Sigma}_{t_\ell}\}_{\ell=1}^{\infty}$ converges to S in the C^1-norm in any compact sphere, and that $\Sigma(t)$ is approximately helicoidal in a neighborhood of each crossing point with multiplicity $(1,0,1,0)$, for t sufficiently close to 0.

q.e.d.

Question 3.1 *Does there exist, for t sufficiently small, an unstable embedded minimal surface bounded by* $\alpha\cup\beta(t)$*, and having genus larger than any compact stable embedded minimal surface bounded by* $\alpha\cup\beta(t)$*?*

The previous theorem described the surfaces $\Sigma(t)$ geometrically. The next theorem describes their topological properties. Let v_1, v_2, e_1, e_2, f_1 and f_2 be defined as in section 1.

Theorem 3.4 *Let* $V\in\mathcal{V}(\mathcal{A},\mathcal{B})$*, and for t sufficiently small let* $\Sigma(V,t)$ *be a minimal surface given in the statement of theorem 3.3. Then the Euler characteristic of* $\Sigma(V,t)$ *is equal to* $(v_1+2v_2)-(e_1+2e_2)+(f_1+2f_2)$*. Since there is exactly one varifold* $V_0\in\mathcal{V}(\mathcal{A},\mathcal{B})$ *with* $v_2 = e_2 = f_2 = 0$*, there is exactly one topological type* $\Sigma(V_0,t)$ *in* $\mathcal{S}(t)$ *with Euler characteristic equal to* $v_1 - e_1 + f_1$*.*

Proof. This follows easily from the description of $\Sigma(t)$, and from the observation that a cell decomposition of $\Sigma(t)$ can be computed via a cell decomposition of the unique varifold V determined by $\alpha\cup\beta$, and corresponding to $\Sigma(t)$. Such a cell decomposition is equivalent to one that has:

— number of vertices equal to $v_1 + 2v_2$.

— number of edges equal to $e_1 + 2e_2$.

— number of faces equal to $f_1 + f_2$.

The unique area-minimizing varifold gives rise to a minimal surface having largest genus, since for this varifold $v_2 = e_2 = f_2 = 0$. Hence the proof is finished.

q.e.d.

In sections 4 and 5 we will prove that the surface $\Sigma(V_0,t)$ is the unique surface of least area in $\mathcal{S}(t)$, for t sufficiently small.

Remark 3.3 *The results in this paper generalize to the case of boundary curves being two collections* $\mathcal{A}$ *and* $\mathcal{B}(t)$ *of disjoint smooth closed Jordan curves contained in the planes* P_0 *and* P_t *respectively. Moreover these results also hold if* $\mathcal{A}$ *and* $\mathcal{B}$ *are contained in the interior of the half plane* ${P'}_0 = \{(x,y,z)\in\mathbb{R}^3| \ \ z=0, x\geq 0\}$*. In this case* $\mathcal{B}(t)$ *is the collection of curves in the plane* ${P'}_t$*, obtained by rotating* P_0 *counterclockwise around the* x*-axis by* $t\frac{\pi}{2}$ *radians.*

The above theorem provides a complete description of the stable surfaces $\Sigma(t)$, description which we will use to prove the uniqueness of the correspondence between $\mathcal{V}(\mathcal{A},\mathcal{B})$ and $\mathcal{S}(t)$ in the next two sections of this paper.

4.0 Proof of stability

The main objective of this section is to prove the following Theorem.

Theorem 4.1 *Let $\Sigma(t)$ be an embedded minimal surface whose boundary is $\alpha\cup\beta(t)$. Suppose that $\Sigma(t)$ is described as in Theorem 3.6. Then, for t sufficiently close to 0, $\Sigma(t)$ is stable.*

4.1 Preliminary lemmas

The proof of theorem 4.1 will follow from some preliminary lemmas. To state these lemmas we will need some additional notation. Let U be a vertical cylindrical neighborhood with height larger than 1 of a helicoidal crossing point. Let $E(t)$ be the intersection of U with $\Sigma(t)$. Suppose that the radius of U is sufficiently small, so that $E(t)$ is stable; note that this can always be accomplished, for an appropriate choice of radius, because although $\Sigma(t)$ is approximately helicoidal in U, the boundary arcs of the helicoidal piece are not parallel, and the multiplicity of the crossing point contained in U is $(0,1,0,1)$; these conditions guarantee that the area of the Gaussian image of $E(t)$, counted with multiplicity, is less than 2π. Let $G(t)$ be one of the connected components of $\hat{\Sigma}(t)$ (defined in Part 1 of the proof of Theorem 3.6) adjacent to $E(t)$, and let $\gamma(t)=\partial E(t)\cap\partial G(t)$. Also recall that here t approaches 0. The lemma that we are about to prove guarantees that the behavior of a supposed Jacobi vector field on $\Sigma(t)$ is not "too wild" away from the crossing points. We remark here, once and for all, that all the Jacobi vector fields we consider are not identically zero, unless explicitly stated otherwise.

Lemma 4.1 *Let $\{\Sigma(t)\}_{t\to 0}$ be a sequence of compact embedded minimal surfaces bounded by $\alpha\cup\beta(t)$, described as in Theorem 3.6, and suppose that all $\Sigma(t)$ are unstable, with Jacobi vector fields given by $u_t:\Sigma(t)\to[0,\infty)$, $u_t=0$ on $\partial\Sigma(t)$, u_t of class C^∞. Then u_t is not a "bump function" on $\gamma(t)$, for t sufficiently close to 0. More precisely, if $p_t\in\gamma(t)$ is a local maximum for the function u_t, then p_t is contained in an arc intersecting $E(t)$ along*

which the values attained by u_t are very close to $u(p)$, of the order of $1-\cos(\vec{N}(p_t),\vec{e}_3)$.

Proof. Since it is not ambiguous, after an appropriate choice of t, we will drop the (t) indicating dependence on t, in this proof. Let us consider the foliation $\{\Sigma + r\vec{e_3}\}_{0\leq r\leq 1}$, and consider the surface Σ' which is obtained by deforming Σ via the Jacobi vector field $\tilde{u}=\frac{u}{\max u}$, namely the set of all the points $q+\tilde{u}(q)\vec{N}(q)$, as q varies in Σ. Let us consider the "restrictions" of the foliation and of the surface Σ' to the subset G of Σ, and let us denote by $\{G_r\}_{0\leq r\leq 1}$ and G' the transformed images of G via the foliation and the Jacobi vector field, respectively. Because all the surfaces we are considering are compact, there exists the maximum value $\overline{r}$ for which the surface G' intersects the foliation $\{P_r\}_{0\leq r\leq 1}$ (after such value the translated surfaces $\{G_r\}_{r\geq\overline{r}}$ are situated "above G'"). By the maximum principle at a boundary point, such last point of contact must be a point $\overline{p}$ contained in the boundary curve $\gamma=(\partial G)\cap(\partial E)$. Let p be the point where $\tilde{u}_{|_G}$ attains its maximum value; since G is an almost horizontal graph, there is no loss of generality in the assumption that the maximum value of $\tilde{u}_{|_G}$ is attained at $\overline{p}$ (in fact this can always be achieved by slightly deforming the curve γ). Let now $S(\eta)$ be a very thin strip which is a η-neighborhood of γ with respect to the metric of Σ, $\eta\approx 0$, and p' a point contained in $(\partial S(\eta)\cap E)\backslash\partial\Sigma$ corresponding to the last intersection point of $\{(P\cup S)_r\}_{0\leq r\leq 1}$ with $(P\cup S(\eta))'$. Let us notice that:

$$1-\overline{r}\leq 1-\cos(\vec{N}(p),\vec{e}_3),$$

and that certainly, if $\overline{r}'$ denotes the value of $0\leq r\leq 1$ corresponding to p', then we have $1-\overline{r}'\leq 1-\overline{r}$, which implies that

$$|(p+\tilde{u}(p)\vec{N}(p))_z-(p'+\tilde{u}(p')\vec{N}(p'))_z|\leq 1-\cos(\vec{N}(p),\vec{e}_3),$$

where $*_z$ denotes the ordinary z-coordinate of a point $*$ in $\mathbb{R}^3$. Now, since the unit normal vector to Σ approaches $\pm\vec{e}_3$ in a continuous fashion away from the crossing points, the above says actually that the difference between $\tilde{u}(p)$ and $\tilde{u}(p')$ is at most $1-\overline{r}+|p_z-{p'}_z|$. Finally, by perturbing γ slightly in all directions around p and applying the above argument to these perturbed curves, the proof of the lemma is complete. q.e.d.

The next lemma is inspired by the paper [12] of N. Kapouleas. It will show that if a Jacobi vector field on a manifold is "close" (in some sense, which will be specified in the statement of the lemma) to a Jacobi vector field on another appropriate manifold, then the first eigenvalues of the Jacobi operator on the two manifolds are also "close" to each other.

Lemma 4.2 *Let U be a neighborhood of a helicoidal crossing point such that the surface $E(t)$ given by the intersection of $\Sigma(t)$ with U is stable, and let $G(t)$ be a connected component of $\hat{\Sigma}(t)$ adjacent to $E(t)$. Let $M_1 = \Sigma(t)$, and let $M_2(\mu)$ be $\Sigma(t)\backslash S(\mu)$, where $S(\mu)$ is a μ-neighborhood of $\gamma = \partial E(t)\cap\partial G(t)$ in the metric of $\Sigma(t)$, $\mu\approx 0$. Let f define a Jacobi vector field on M_1. Suppose that it is possible to deform the function $f\colon M_1 \to \mathbb{R}$ to obtain a Jacobi vector field $G(f)\colon M_2(\mu) \to \mathbb{R}$, which is zero on $\partial M_2(\mu)\cap\partial\Sigma$ and such that it satisfies the three additional conditions:*

(i) $\|f\|_\infty \leq 2\|G(f)\|_\infty$;

(ii) $|\langle f, f\rangle_2 - \langle G(f), G(f)\rangle_2| \leq \epsilon\|f\|_\infty\|f\|_\infty$;

(iii) $\|\nabla(G(f))\|_2 \leq (1+\epsilon)\|\nabla f\|_2 + \epsilon\|f\|_\infty$.

Then if ϵ can be made arbitrarily small, the first eigenvalue of the Jacobi operator on M_1 can be made arbitrarily close to the first eigenvalue of the Jacobi operator on $M_2(\mu)$.

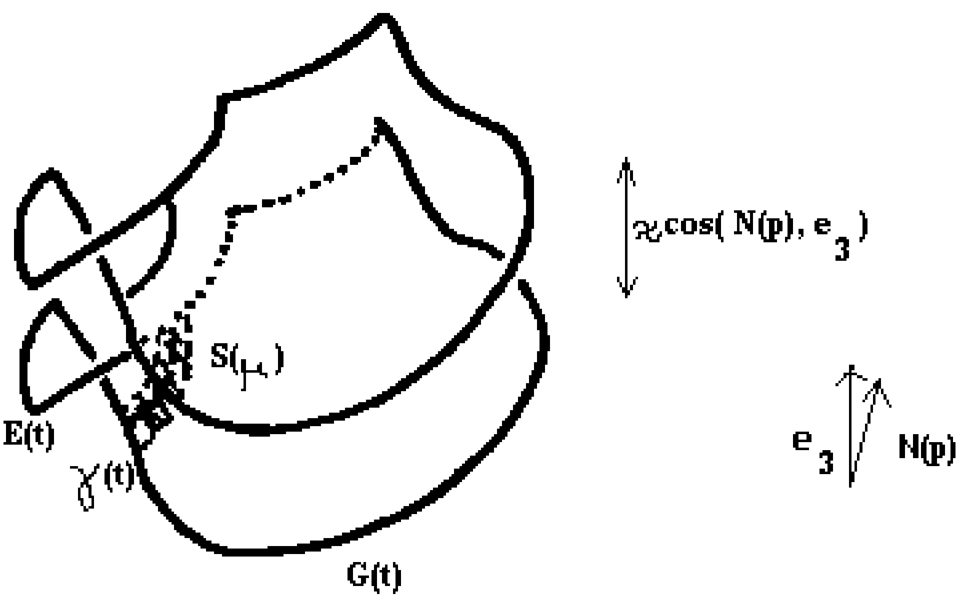

Figure 4.3: Illustration of the proof of the stability theorem 4.1.

Proof. Let us consider

$$\lambda_1(M_1) = \inf_{\overline{f}\in C_0^\infty(M_1)} \frac{\|\nabla\overline{f}\|_2^2}{\|\overline{f}\|_2^2}.$$

Then for each $\epsilon > 0$, if $f\in C_0^\infty(M_1)$ is a Jacobi vector field, one has, as observed in paragraph 1.2,

$$\|\nabla f\|_2^2 < (\lambda_1(M_1) + \epsilon)\|f\|_2^2.$$

Let us choose $\epsilon = \sqrt{Area(S(\mu))}$. Because of our hypotheses, f induces a $G(f)\in C_0^\infty(M_2(\mu))$ such that condition (ii) above is satisfied, namely

$$|\, \|f\|_2^2 - \|G(f)\|_2^2 \,|\leq\epsilon\|f\|_\infty,$$

which implies

$$\|\nabla f\|_2^2 < (\lambda_1(M_1) + \epsilon)(\|G(f)\|_2 + \epsilon\|f\|_\infty)^2.$$

Moreover condition (iii) above implies

$$(\frac{\|\nabla G(f)\|_2 - \epsilon\|f\|_\infty}{1+\epsilon})^2\leq\|\nabla f\|_2^2,$$

which yields the conclusion:

$$(\frac{\|\nabla G(f)\|_2 - \epsilon\|f\|_\infty}{1+\epsilon})^2 < (\lambda_1(M_1) + \epsilon)(\|G(f)\|_2 + \epsilon\|f\|_\infty)^2.$$

If ϵ can be chosen arbitrarily small at the beginning, when $\epsilon \to 0$ one has

$$\|\nabla G(f)\|_2^2\leq\lambda_1(M_1)\|G(f)\|_2^2,$$

namely $\lambda_1(M_2(\mu))\leq\lambda_1(M_1)$. Clearly in our case we can exchange the roles of M_1 and $M_2(\mu)$, since $M_2(\mu)$ is contained in M_1, and have that as $\mu \to 0$, $\lambda_1(M_1) \to \lambda_1(M_2(\mu))$, and vice-versa. In particular, we have that if M_1 is unstable, so is $M_2(\mu)$, and vice-versa. q.e.d.

In the next lemma we will show that, for the surfaces M_1 and $M_2(\mu)$ defined above, the construction of $G(f)$ having the properties required in Lemma 4.2 is possible.

Lemma 4.3 *Let $M_1 = \Sigma(t)$, and $M_2(\mu) = \Sigma(t)\backslash S(\mu)$, $\mu\approx 0$, and suppose that $f\in C_0^\infty(M_1)$ defines a Jacobi vector field on M_1. Then it is possible to define a not identically zero function $G(f)\in C_0^\infty(M_2(\mu))$, in such a way that the conditions (i), (ii), (iii) of Lemma* 3.3 *are satisfied.*

Proof. Let $f \in C_0^\infty(M_1)$ be the Jacobi vector field on M_1, and without loss of generality let us suppose that the maximum of f is attained at some point belonging to γ (otherwise the proof of the lemma is still valid, as one can easily see: this hypothesis takes care of the "worst possible case"). Moreover let ϕ be a bump function of class C^∞ on a thin strip S containing γ and having area less than δ such that ϕ is constantly equal to 1 on $M_1 \backslash S$, constantly equal to zero on $S' \subset S$ (S' is a strip containing γ and contained in S), and such that $|\nabla\phi| \le \frac{2}{\delta^{1/4}}$. Define now $G(f) = \phi f$. We get:

(i) $\|f\|_\infty < 2\|G(f)\|_\infty$, because of the property shown in lemma 3.2.

(ii) $|\langle f, f\rangle - \langle G(f), G(f)\rangle| \le \delta^2 \|f\|_\infty \|f\|_\infty$, namely

$$
\begin{aligned}
|\|f\|_2^2 - \|\phi f\|_2^2| &= |\int_{E\cup P} f^2 - \int_{E\cup P} \phi^2 f^2| \\
&\le \int_S |f^2 - \phi^2 f^2| \\
&= \int_S |f^2||1-\phi^2| \\
&\le \|f^2\|_\infty \delta^2 \|1-\phi^2\|_\infty \\
&\le \delta^2 \|f\|_\infty^2.
\end{aligned}
$$

(iii) $\|\nabla(G(f))\|_2 \le (1+\epsilon)\|\nabla f\|_2 + \epsilon\|f\|_\infty$, since

$$
\begin{aligned}
\|\nabla(\phi f)\|_2^2 &= \int_{M_2} |\nabla(\phi f)|^2 \\
&\leq \int_{M_2} (|\nabla\phi||f| + |\phi||\nabla f|)^2 \\
&\leq \int_S |\nabla\phi|^2|f|^2 + 2\int_S |\nabla\phi||f||\nabla f| + \int_{M_2} |\nabla f|^2 \\
&\leq \|(\nabla\phi)^2(f)^2\|_1 + 2\frac{2}{\delta^{\frac{1}{4}}}\int_S |f||\nabla f| + \int_{M_2} |\nabla f|^2 \\
&\leq \|(\nabla\phi)^2\|_2\|f^2\|_2 + \frac{4}{\delta^{\frac{1}{4}}}\|(f)(\nabla f)\|_1 + \|\nabla f\|_2^2 \\
&\leq (\int_S |\nabla\phi|^4)^{\frac{1}{2}}(\int_S |f|^4)^{\frac{1}{2}} + \frac{4}{\delta^{\frac{1}{4}}}\|f\|_2\|\nabla f\|_2 + \|\nabla f\|_2^2 \\
&\leq (\delta^2\frac{16}{\delta})^{\frac{1}{2}}(\delta^2\|f\|_\infty^4)^{\frac{1}{2}} + \frac{4}{\delta^{\frac{1}{4}}}\|f\|_2\|\nabla f\|_2 + \|\nabla f\|_2^2 \\
&\leq (16\delta)^{\frac{1}{2}}(\delta^2\|f\|_\infty^4)^{\frac{1}{2}} + \frac{4}{\delta^{\frac{1}{4}}}(\delta^2\|f\|_\infty^2)^{\frac{1}{2}}\|\nabla f\|_2 + \|\nabla f\|_2^2 \\
&= 4\delta^{\frac{3}{2}}\|f\|_\infty^2 + \frac{4\delta}{\delta^{\frac{1}{4}}}\|f\|_\infty\|\nabla f\|_2 + \|\nabla f\|_2^2 \\
&= (2\delta^{\frac{3}{4}}\|f\|_\infty + \|\nabla f\|_2)^2.
\end{aligned}
$$

The assertion hence follows by choosing $\epsilon \leq \min\{\delta^2, 2\delta^{\frac{3}{4}}\}$. q.e.d.

In the proof of theorem 4.1 we will denote, with the same notation adopted previously, $\gamma = \partial E_{t_n} \cap \partial P_{t_n}$. We will prove the theorem here by taking $M_1 = \Sigma_{t_n}$, and $M_2(\mu) = \Sigma_{t_n} \backslash S(\mu)$, which we suppose to be stable. However, since the number of crossing points is finite, the proof also holds if we take $M_2(\mu_1, \dots, \mu_{n'}) = \Sigma_{t_n} \backslash \bigcup_{J=1}^{n'} S(\mu_j)$, $n' \leq n$, which we assume to be stable, and $S(\mu_j) = \partial(E_{t_n}(j)) \cap \partial(P_{t_n})$, where $E_{t_n}(j)$ is a sufficiently small neighborhood of the crossing point p_j in Σ_{t_n}.

Let us now give a proof of theorem 4.1, which we restate for easy reference.

Theorem 4.1. Let $\Sigma(t)$ be an embedded minimal surface with boundary $\alpha \cup \beta(t)$. Suppose that $\Sigma(t)$ is described as in theorem 3.6. Then, for t sufficiently close to 0, $\Sigma(t)$ is stable.

Proof. Suppose that the assertion stated in the theorem is false. Then there would exist a sequence $t_n \to 0$, corresponding to which there would be a sequence of unstable minimal surfaces $\Sigma(t_n)$ with boundary $\alpha \cup \beta(t_n)$, and

described according to theorem 3.6. Hence there would exist a sequence of Jacobi vector fields f_{t_n} defined on $\Sigma(t_n)$, all having the property proven in lemma 4.2. Therefore, by induction and by lemmas 4.2, 4.3, 4.4, it would be possible to fix a positive μ such that in a μ-neighborhood (strip) of γ one could define a bump function having bounded gradient which, when multiplied by f_{t_n} would yield a new function $G(f_{t_n})$ defined on the stable part given by $M_2(\mu)$ (the fact that this would be possible for each n follows from lemma 3.2). But the Rayleigh quotient associated to such a function can be made (dependently on μ) arbitrarily close to a number which is strictly less than 2, thus producing a contradiction. For the sake of completeness, let us notice here that lemma 4.2 is of fundamental importance in this proof, because it ensures that none of the functions $G(f_n)$ is the function identically equal to zero. q.e.d.

5.0 Proof of the Main Theorem

In this section we will put together the facts proven so far, to give a proof of the main theorem, and to give the exact number of $\Sigma(V,t)$ for t sufficiently small.

5.1 Uniqueness

Let $Slab(t)$ be the slab determined by the planes P_0 and P_t. Let $\mathcal{V}(\alpha,\beta)$ be the finite collection of varifolds determined by $\alpha\cup\beta$. By the results proved in the previous sections, we know that for all values of t sufficiently small, each varifold V in $\mathcal{V}(\alpha,\beta)$ determines at least one compact stable embedded minimal surface $\Sigma(V,t)$ in $\mathcal{S}(t)$ bounded by $\alpha\cup\beta(t)$. We now show:

Theorem 5.1 *The natural correspondence between $\mathcal{V}(\alpha,\beta)$ and $\mathcal{S}(t)$ is a well defined and one-to-one correspondence, in the sense that to each varifold in $\mathcal{V}(\alpha,\beta)$ corresponds one and only one minimal surface in $\mathcal{S}(t)$, for t sufficiently small.*

Proof. The theorem will be proven by contradiction.

Let $\{\overline{\Sigma^1}(t) \colon 0\leq t\leq t^{\#}\}$ and $\{\overline{\Sigma^2}(t) \colon 0\leq t\leq t^{\#}\}$ be two distinct families (for $t^{\#}$ sufficiently close to 0) of minimal surfaces with the same boundary $\alpha\cup\beta(t)$, existing for all $0\leq t\leq t^{\#}$, and having the same limit varifold V. Let $M_1(t)$ be the unbounded connected component of the region T of space given by the collection of points in the slab which are "outside" of $\overline{\Sigma^1}(t)\cup\overline{\Sigma^2}(t)$,

R be the union of the bounded connected regions given by the points "in between" $\overline{\Sigma^1}(t)$ and $\overline{\Sigma^2}(t)$, and $M_2(t)$ be the closure of $T\backslash R$. Notice that $M_1(t)$ contains the truncated cylinder above $P\backslash V$, and that $M_2(t)$ is contained in a small neighborhood of the truncated cylinder above V. Notice that $\partial(M_1 \cup M_2)$ strictly contains $\overline{\Sigma^1}(t)\cup\overline{\Sigma^2}(t)$. Since $\overline{\Sigma^1}(t)$ and $\overline{\Sigma^2}(t)$ are stable by theorem 3.1, applying the existence theorem by Meeks and Yau stated in section 3 to M_1, we obtain the existence of a stable minimal surface $\Sigma_a(t)$ "above" $\overline{\Sigma^1}(t)\cup\overline{\Sigma^2}(t)$. By the same theorem of Meeks and Yau, applied to the region M_2, we obtain the existence of a stable minimal surface $\Sigma_b(t)$ "below" $\overline{\Sigma^1}(t)\cup\overline{\Sigma^2}(t)$. By construction, $\Sigma_a(t)$ and $\Sigma_b(t)$ are disjoint from each other. Moreover, for t sufficiently close to 0, the two stable minimal surfaces $\Sigma_a(t)$ and $\Sigma_b(t)$ obtained in this way can be described as stated in theorem 2.5, namely as approximately helicoidal around the crossing points with multiplicity $(0,1,0,1)$, and as almost horizontal graphs away from the crossing points with multiplicity $(0,1,0,1)$, by virtue of theorem 4.1. Hence $\Sigma_a(t)$ and $\Sigma_b(t)$ are homeomorphic to the $\overline{\Sigma^i}(t)$, and furthermore $\Sigma_a(t)$ and $\Sigma_b(t)$ are normal graphs above $\overline{\Sigma^i}(t)$, and hence over each other, namely there exists a function $h\in C_0^\infty(\Sigma_b(t))$ such that:

$$\Sigma_a(t) = \Sigma_b(t) + h\vec{N}(\Sigma_b(t)), \tag{5.3}$$

that is, for each $q\in\Sigma_a(t)$ there exists a unique $q'\in\Sigma_b(t)$ such that

$$q = q' + h(q')\vec{N}(q'),$$

where $\vec{N}(q')$ is the normal vector to $\Sigma_b(t)$ in q'. Hence $\Sigma_a(t)$ and $\Sigma_b(t)$ converge to the same limiting varifold, as $t\to 0$. It also follows from (5.1) that $\Sigma_a(t)\cup\Sigma_b(t)$ bounds a product region, say $\mathcal{R}(t)$. Since $\Sigma_a(t)$ and $\Sigma_b(t)$ are normal graphs over each other, we know that the angle between the two normal vectors to $\Sigma_a(t)$ and $\Sigma_b(t)$ at a boundary point is strictly between 0 and π. Then by the minimax theorems due to Pitts and Rubinstein [30], and generalized by Jost [11] to the case of nonempty boundary, $\alpha\cup\beta(t)$ is also the boundary of an unstable embedded minimal surface $\Sigma^*(t)$, contained in $\mathcal{R}$, for all t sufficiently small. We now wish to show that $\Sigma^*(t)$ is homeomorphic to $\Sigma_a(t)$ and $\Sigma_b(t)$, and that $\Sigma^*(t)$ actually has the same geometric description as $\Sigma_a(t)$ and $\Sigma_b(t)$, which will imply, for example, that $\lim_{t\to 0}\Sigma^*(t) = \lim_{t\to 0}\Sigma_a(t) = \lim_{t\to 0}\Sigma_b(t) = V$. First notice that the connected components of the complement in $\Sigma^*(t)$ of the union of small neighborhoods of the crossing points, is the union of almost horizontal graphs, just like $\Sigma_a(t)$ and $\Sigma_b(t)$. In fact, the area of each such connected component is almost equal to the area of the corresponding components of $\Sigma_a(t)$

and $\Sigma_b(t)$. The proof of this fact follows from comparing the area of $\Sigma^*(t)$ with the maximum area of the family $\Sigma_a(t) = \Sigma_b(t) + th\vec{N}(\Sigma_b(t))$, indexed by $t\in[0,1]$. The definition of minimax implies that the area of $\Sigma^*(t)$ cannot be larger than the maximum area of the family; but since every surface in the family is a graph over $\Sigma_b(t)$, we can estimate the area of $\Sigma^*(t)$ by that of $\Sigma_a(t)$ and $\Sigma_b(t)$, and the proof of the claim follows as in part 1 of the proof of theorem 3.6, since the area estimates just proven allow us to apply R. Schoen's curvature estimates. So we know that $\Sigma^*(t)$ given by very flat graphs away from the crossing points. In the neighborhood of a crossing point with multiplicity $(0,1,0,1)$, consider the part of $\Sigma^*(t)$ bounded by four geodesic arcs, constructed as in part 3 of the proof of theorem 3.6. Such quadrilateral region has sum of the external angles strictly less than 4π. By Nitsche's 4π-theorem stated in section 2, applied to an analytic smoothing having total curvature less than 4π of the quadrilateral defined by the geodesic arcs, we know that this quadrilateral region must bound a *stable* minimal surface which is topologically a disk. This means that $\Sigma^*(t)$ can be described as in theorem 3.6, for all t sufficiently small, and that $\Sigma^*(t)$ is a graph over $\Sigma_b(t)$. But then, for t sufficiently close to 0, $\Sigma^*(t)$ must be stable, by theorem 4.1. This produces a contradiction and finishes the proof of the uniqueness theorem. q.e.d.

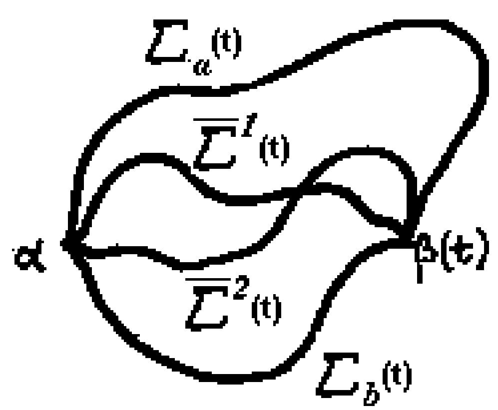

Figure 5.4: Illustration of the proof of the uniqueness theorem 5.1.

5.2 A bound on the number of $\Sigma(V,t)$'s

In this section we will derive a formula which gives an upper bound for the number of compact stable minimal surfaces bounded by a finite number of Jordan curves in close planes of $\mathbb{R}^3$.

By theorem 5.1, we know that once the multiplicity of the limiting varifold V is fixed, then there is a unique stable compact minimal surface bounded by $\alpha_t \cup \beta$, for t sufficiently close to 0. So the number of stable compact embedded minimal surfaces will be determined once we are able to express exactly how many are the possible multiplicities for the limiting varifold V. In order to understand this, let us assign to each connected component of $P_0 \backslash (\alpha \cup \beta)$ the sign "+" or "−" in such a way that the unbounded component C_u is given the "−" sign and adjacent components have opposite signs. Moreover notice that $P_0 \backslash (\alpha \cup \beta)$ determines a finite number of varifolds, by assigning to each of its connected components one of the numbers 0, 1 or 2 (note that the connected components having the "+" sign can only be assigned multiplicity one).

So the multiplicity is totally determined for the components of V with the "+" sign. The only places where there are different multiplicities allowed are the components of V with the "−" sign. Of course we only have two choices for the multiplicity of such components: *zero* or *two*. However we are not free to assign multiplicities arbitrarily, as $(1,2,1,2)$ crossing points must be avoided. Let f_-^i be the number of components of V having "−" sign, and contained inside $\mathcal{R}(\alpha) \cap \mathcal{R}(\beta)$, and let f_-^o be the number of components of V having "−" sign, and contained outside $\mathcal{R}(\alpha) \cup \mathcal{R}(\beta)$. Then the above observations translate in the following

Corollary 5.1 *Once the limiting cycle Z, is given, the number of stable compact minimal surfaces $\Sigma(t)$ such that $\partial(\Sigma(t)) \to Z$ as $t \to 0$, is bounded above by*

$$2^{f_-^i} + 2^{f_-^o}.$$

Remark 5.1 *It would be interesting to get similar bounds on the number of unstable embedded minimal surfaces bounded by Jordan curves in close planes.*

6.0 A nonexistence result

In this section we observe that our main theorem provides some evidence in support of a conjecture made by W. Meeks in [14]. The conjecture states that there are no minimal surfaces of positive genus bounded by two convex curves in parallel planes of $\mathbb{R}^3$. One consequence of our main theorem is:

Corollary 6.1 *There exist no stable minimal surfaces of positive genus bounded by two convex curves in parallel planes of* $\mathbb{R}^3$*, when the distance between the two planes is sufficiently small (or if the two planes are not parallel, but they intersect at a sufficiently small angle).*

Previous results by R. Schoen [32], W. Meeks and B. White [18] [19] [20], and earlier M. Shiffman [34], also supported evidence towards Meeks's conjecture.

Remark 6.1 *If one could show that as* t*, the distance between the planes, increases, the number of* stable *minimal surfaces bounded by two convex curves does not increase, Meeks's conjecture would follow, at least in the stable case.*

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University of California,
Santa Cruz